

RESULTS ON THE REGULARITY OF SQUARE-FREE MONOMIAL IDEALS

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ABSTRACT. In a 2008 paper, the first author and Van Tuyl proved that the regularity of the edge ideal of a graph G is at most one greater than the matching number of G . In this note, we provide a generalization of this result to any square-free monomial ideal. We define a 2-collage in a simple hypergraph to be a collection of edges with the property that for any edge E of the hypergraph, there exists an edge F in the collage such that $|E \setminus F| \leq 1$. The Castelnuovo-Mumford regularity of the edge ideal of a simple hypergraph is bounded above by a multiple of the minimum size of a 2-collage. We also give a recursive formula to compute the regularity of a vertex-decomposable hypergraph. Finally, we show that regularity in the graph case is bounded by a certain statistic based on maximal packings of nondegenerate star subgraphs.

1. INTRODUCTION

Let k be a field. There is a natural correspondence between square-free monomial ideals in $R = k[x_1, \dots, x_n]$ and simple hypergraphs over the vertices $V = \{x_1, \dots, x_n\}$. This correspondence has evolved to be an instrumental tool in an active research program in combinatorial commutative algebra — we recommend any of [12, 17, 19, 23] for an overview. One goal of this research program is to relate algebraic properties and invariants of a square-free monomial ideal to combinatorial properties and statistics of the corresponding simple hypergraph. In this note, we will examine the Castelnuovo-Mumford regularity of such ideals, which has been previously studied in work including [5, 7, 9, 14, 16, 20, 22].

In [10, Theorem 6.7], the first author and Van Tuyl showed that the regularity of the edge ideal of a graph G is at most one greater than the matching number of G . Indeed, it follows from their proof and was explicitly noticed in [25] that the regularity of the edge ideal of a graph is at most one greater than the minimum size of a maximal matching. The first goal of this note is extend this result to the edge ideal of a hypergraph, i.e., to any square-free monomial ideal.

Our first bound for the Castelnuovo-Mumford regularity of a square-free monomial ideal is based on the notion of 2-collage. If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, then a *2-collage* for \mathcal{H} is a subset \mathcal{C} of the edges with the property that for each $E \in \mathcal{E}$ we can delete a vertex v so that $E \setminus \{v\}$ is contained in some edge of \mathcal{C} . For uniform hypergraphs, the condition for a collection \mathcal{C} of the edges to be a 2-collage is equivalent to requiring that for any edge E not in \mathcal{C} , there exists $F \in \mathcal{C}$ such that the symmetric difference of E and F consists of

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exactly two vertices. When \mathcal{H} is a graph, the notion of 2-collage is exactly that of a maximal matching. Our first main result is:

Theorem 1.1. *Let \mathcal{H} be a simple d -uniform hypergraph with edge ideal $I \subseteq R$, and let c be the minimum size of a 2-collage in \mathcal{H} . Then $\text{reg}(R/I) \leq (d-1)c$.*

Indeed, Theorem 1.1 will follow from the following more general result:

Theorem 1.2. *Let \mathcal{H} be a simple hypergraph with edge ideal $I \subseteq R$, and let $\{E_1, \dots, E_c\}$ be a 2-collage in \mathcal{H} . Then $\text{reg}(R/I) \leq \sum_{i=1}^c (|E_i| - 1)$.*

In the case where our hypergraph \mathcal{H} is not a graph, a minimum 2-collage in \mathcal{H} is bigger than the *minimax matching number* (that is, the minimum size of a maximal matching). We shall see in Example 3.5 that even in the uniform case, the bound in Theorem 1.1 is no longer true if we replace c by the minimax matching number m of \mathcal{H} , and in fact that $\text{reg}(R/I)$ can be arbitrary larger than $(d-1)m$. If \mathcal{H} is a graph, the minimum size of a 2-collage is easily seen to be the minimax matching number, so Theorem 1.1 restricted to graphs recovers [10, Theorem 6.7] and [25, Theorem 11 and discussion following].

Upper bounds are most interesting when compared with lower bounds, and while hypergraph matchings do not in general seem to give any upper bound for regularity, a related notion will give a lower bound. We call a collection $\{E_1, \dots, E_\ell\}$ of edges in \mathcal{H} an *induced matching* if they form a matching in \mathcal{H} (i.e., they are pairwise disjoint), and they are exactly the edges of the induced subhypergraph of \mathcal{H} over the vertices contained in $\bigcup_{i=1}^\ell E_i$. The *induced matching number* of \mathcal{H} , denoted by $\nu_{\text{ind}}(\mathcal{H})$, is the maximum size of an induced matching in \mathcal{H} . The following was proved in [10, Theorem 6.5] for properly connected simple hypergraphs, and was extended to all simple hypergraphs in [19, Corollary 3.9]:

Theorem 1.3. [10, Theorem 6.5][19, Corollary 3.9] *Let \mathcal{H} be a simple hypergraph with edge ideal $I \subseteq R$, and let $\{E_1, \dots, E_\ell\}$ be an induced matching in \mathcal{H} . Then $\text{reg}(R/I) \geq \sum_{i=1}^\ell (|E_i| - 1)$.*

For comparison, we restate in a form parallel to Theorem 1.1:

Theorem 1.4. *Let \mathcal{H} be a simple d -uniform hypergraph with edge ideal $I \subseteq R$. Then $\text{reg}(R/I) \geq (d-1)\nu_{\text{ind}}(\mathcal{H})$.*

A second goal of this note is to describe the regularity of *vertex-decomposable* graphs, a class of graphs that has garnered considerable recent attention [22, 26]. In particular, the quotient ring associated to the edge ideal of a vertex-decomposable graph or hypergraph is sequentially Cohen-Macaulay [4]. We give a recursive formula to compute regularity of any vertex-decomposable hypergraph:

Theorem 1.5. *Let \mathcal{H} be a vertex-decomposable simple hypergraph with edge ideal $I \subseteq R$, and with v the initial vertex in the shedding order. Then $\text{reg}(R/I) = \max \{\text{reg}(I : v) + 1, \text{reg}(I, v)\}$.*

We will actually prove a slightly more general (if somewhat technical) result, which weakens the vertex-decomposability condition to a sequentially Cohen-Macaulay condition on the vertex deletion subcomplex. We state this precisely as Theorem 4.2 below.

Note that Dao, Huneke and Schweig observed [7] that $\text{reg}(I)$ is equal to one of $\text{reg}(I : v) + 1$ or $\text{reg}(I, v)$ for any hypergraph \mathcal{H} and vertex v . Our result is that $\text{reg}(I)$ is always the larger of the two in the case of a vertex-decomposable hypergraph and its shedding vertex.

Our third goal will be to give a new upper bound for the regularity of any graph. Our upper bound will be based on a certain packing-type invariant, as follows. The *closed neighborhood* of x in a graph G , denoted $N_G[x]$, is the subset of vertices consisting of x and all of its neighbors. A closely related notion is the *star* at x , which is the subgraph on $N_G[x]$ with edge set consisting of all edges of G incident to x . We say that a star is *nondegenerate* if $\deg x > 1$, so that the star doesn't consist of a single vertex or single edge.

Our upper bound will be based on packing nondegenerate stars into G . We say a set of stars is *center-separated* if the center of a star is not contained in any other star. Thus, a set of center-separated nondegenerate stars corresponds to an independent set consisting of vertices with degree greater than 1. After deleting the vertices of the stars in a maximal center-separated star packing \mathcal{P} , an induced matching of G will remain. Let $\zeta_{\mathcal{P}}$ be the number of stars in the packing plus the number of edges in the induced matching remainder, and let $\zeta(G)$ be the maximum $\zeta_{\mathcal{P}}$ over all maximal center-separated packings of nondegenerate stars. Our third main theorem is:

Theorem 1.6. *Let G be a graph with edge ideal $I \subseteq R$. Then $\text{reg}(R/I) \leq \zeta(G)$.*

It is clear that $\zeta(G)$ is at most the matching number of G , so Theorem 1.6 is another generalization of the matching upper bound of Hà and Van Tuyl. Theorem 1.6 also improves on bounds of Moradi and Kiani [18] proved with an additional assumption of vertex-decomposability and/or shellability, as we will discuss in Remarks 5.2 and 5.6. Theorem 1.6 is proved inductively, and the main step in the induction (Lemma 5.5) may be of independent interest.

This paper is organized as follows. In Section 2, we shall collect the necessary notations and terminology. The proof of Theorem 1.2 is given in Section 3. The main tool for this theorem is a result of Kalai and Meshulam [13] bounding the regularity of the sum of square-free monomial ideals. In Section 4, we use the Stanley-Reisner face ring correspondence and the combinatorial topology of simplicial complexes to prove Theorem 1.5. Finally, in Section 5, we prove Theorem 1.6.

2. NOTATION AND TERMINOLOGY

2.1. Hypergraphs and edge ideals. A *hypergraph* \mathcal{H} consists of a set $V = \{x_1, \dots, x_n\}$, called *vertices*; and a collection \mathcal{E} of nonempty subsets of V , called *edges*. We will use $V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ to denote the sets of vertices and edges, respectively, of \mathcal{H} . A hypergraph is *simple* if there are no nontrivial containments among the edges (i.e., if $E \subseteq E'$ are edges then $E = E'$). All hypergraphs discussed in this paper will be simple. Simple hypergraphs have been studied under several other names, including “clutter” and “Sperner system”. An important family of simple hypergraphs are *d-uniform hypergraphs*, in which every edge contains exactly d vertices.

Let k be a field, and identify the vertices in V with the variables in a polynomial ring $R = k[x_1, \dots, x_n]$. The following construction gives a one-to-one correspondence between square-free monomial ideals in R and simple hypergraphs over V :

Definition 2.1. Let $\mathcal{H} = (V, \mathcal{E})$ be a simple hypergraph. The *edge ideal* of \mathcal{H} is the square-free monomial ideal

$$I(\mathcal{H}) = (x^E = \prod_{x \in E} x \mid E \in \mathcal{E}) \subseteq R.$$

Certain substructures of a hypergraph will be important to us. If \mathcal{H} is a hypergraph with an edge E , then $\mathcal{H} \setminus E$ will denote the hypergraph obtained from \mathcal{H} by removing E from the edge set. The *induced subhypergraph* of \mathcal{H} on a subset W of the vertex set is the hypergraph over vertex set W with edge set consisting of all edges of \mathcal{H} that are contained in W .

2.2. Simplicial complexes and vertex-decomposability. The edge ideal $I(\mathcal{H})$ is a square-free monomial ideal, so it can also be viewed as the Stanley-Reisner ideal of a simplicial complex as follows: We call a collection \mathcal{B} of the vertices an *independent set* if there is no edge E in \mathcal{H} such that $E \subseteq \mathcal{B}$. The *independence complex* of \mathcal{H} , denoted by $\Delta(\mathcal{H})$, is the simplicial complex whose faces are independent sets in \mathcal{H} . It is immediate from the definitions that if I_Δ denotes the Stanley-Reisner ideal of Δ , then $I(\mathcal{H}) = I_{\Delta(\mathcal{H})}$.

If v is a vertex of the simplicial complex Δ , then the *deletion* of v from Δ , denoted by $\text{del}_\Delta(v)$, is the simplicial complex over the vertex set $V \setminus \{v\}$ with faces $\{\sigma \mid \sigma \in \Delta, v \notin \sigma\}$. An *induced subcomplex* of Δ is obtained by (successively) deleting a set of vertices. The *link* of v in Δ , denoted by $\text{link}_\Delta v$, is the subcomplex of $\text{del}_\Delta v$ with faces $\{\sigma \mid \sigma \in \text{del}_\Delta v, v \cup \sigma \in \Delta\}$.

A simplicial complex Δ is recursively defined to be *vertex-decomposable* if either

- (a) Δ is a simplex, or
- (b) there exists a vertex v such that both $\text{del}_\Delta(v)$ and $\text{link}_\Delta(v)$ are vertex-decomposable, and the facets of $\text{del}_\Delta(v)$ are facets of Δ .

A vertex satisfying the condition in (b) is called a *shedding vertex*, and the recursive choice of vertices is called a *shedding order*. When it causes no confusion, we will call a simple hypergraph \mathcal{H} *vertex-decomposable* if its independence complex $\Delta(\mathcal{H})$ is vertex-decomposable.

A complex is *shellable* if there is an ordering of its facets obeying certain restrictions, the precise details of which will not be important for us. It is well-known that

$$\Delta \text{ vertex-decomposable} \implies \Delta \text{ shellable} \implies \Delta \text{ sequentially Cohen-Macaulay}.$$

For additional background on the combinatorics of simplicial complexes, including vertex-decomposability and shellability, we refer to e.g. [2, 15]; for background on the connection with commutative algebra, we refer to [17, 21].

2.3. Regularity. Recall that the *Castelnuovo-Mumford regularity* (or just *regularity*) of an R -module M can be defined as

$$\operatorname{reg}(M) = \max_i \{ \max\{j \mid \operatorname{Tor}_i^R(M, k)_j \neq 0\} - i \}.$$

For an overview of and background on Castelnuovo-Mumford regularity, we refer to the recent survey article [6].

We observe that for R a polynomial ring, $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$; thus, it is equivalent to study the regularity of the edge ideal or the corresponding quotient ring. Our notation is a bit careless about what polynomial ring we are working over: this is justified, as if S is any polynomial ring over k containing R (with additional variables not appearing in I), then $\operatorname{reg}(R/I) = \operatorname{reg}(S/I)$. Of our main theorems, only Theorem 4.2 depends on the choice of the field k , and this only insofar as the sequentially Cohen-Macaulay property may depend on k .

In Section 4, we will find it helpful to work with regularity through independence complexes and the Stanley-Reisner correspondence. We summarize the connection:

Lemma 2.2. *For a simplicial complex Δ , the following are equivalent:*

- (1) $\operatorname{reg}(R/I_\Delta) \geq d$.
- (2) $\tilde{H}_{d-1}(\Delta[S]) \neq 0$, where $\Delta[S]$ denotes the induced subcomplex on some subset S of vertices.
- (3) $\tilde{H}_{d-1}(\operatorname{link}_\Delta \sigma) \neq 0$ for some face σ of Δ .

We briefly sketch a proof: The equivalence of (1) and (2) follows directly from the Betti number characterization of regularity [6], together with Hochster's formula (as stated in [17, Corollary 5.12]). The equivalence of (1) and (3) follows directly from the local cohomology characterization of regularity [6], together with the fact that $H_{\mathfrak{m}}^i(R/I_\Delta)_{-\sigma} \cong \tilde{H}^{i-|\sigma|-1}(\operatorname{link}_\Delta \sigma)$ [17, Chapter 13.2].

Kalai and Meshulam [13, Proposition 3.1] have also given a direct proof of the equivalence of (2) and (3).

Remark 2.3. In some sources in the topological combinatorics literature [1, 13], complexes Δ with $\operatorname{reg}(R/I_\Delta) \geq d$ are called *d-Leray*, and $\operatorname{reg}(R/I_\Delta)$ is referred to as the *Leray number* of Δ .

The following well-known lemma follows directly from characterization (2) of Lemma 2.2, and tells us that regularity may be regarded as giving a measure of the complexity of \mathcal{H} .

Lemma 2.4. *Let \mathcal{H} be a simple hypergraph. Then $\operatorname{reg} I(\mathcal{H}) \geq \operatorname{reg} I(\mathcal{H}')$ for any induced subhypergraph \mathcal{H}' of \mathcal{H} .*

A similar result holds for links:

Lemma 2.5. *Let Δ be a simplicial complex. Then $\operatorname{reg}(R/I_\Delta) \geq \operatorname{reg}(R/I_{\operatorname{link}_\Delta \sigma})$ for any face σ of Δ .*

3. REGULARITY AND COLLAGES

In this section, we prove Theorem 1.2, bounding the regularity of a square-free monomial ideal with a collage. For completeness, and because the proof is short, we begin by proving the bound from below of Theorem 1.3:

Proof of Theorem 1.3. Let $\{E_1, \dots, E_\ell\}$ be an induced matching in \mathcal{H} and let \mathcal{H}' be the induced subhypergraph of \mathcal{H} over the vertices contained in $\bigcup_{i=1}^\ell E_i$. Since the E_i 's are pairwise disjoint, the regularities add, so we have $\text{reg}(R/I(\mathcal{H}')) = \sum_{i=1}^\ell (|E_i| - 1)$, and the result follows by Lemma 2.4. \square

In our proof of Theorem 1.2, we shall make use of the following theorem of Kalai and Meshulam.

Theorem 3.1. [13, Theorem 1.4] *Let \mathcal{H} and $\mathcal{H}_1, \dots, \mathcal{H}_s$ be simple hypergraphs over the same vertex set V such that $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^s \mathcal{E}(\mathcal{H}_i)$. Then*

$$\text{reg}(R/I(\mathcal{H})) \leq \sum_{i=1}^s \text{reg}(R/I(\mathcal{H}_i)).$$

Remark 3.2. Kalai and Meshulam gave a topological proof of Theorem 3.1 via the correspondence in Lemma 2.2. Theorem 3.1 was later extended to arbitrary (not necessarily square-free) monomial ideals by Herzog [11], who used algebraic techniques.

We will need a technical lemma. If E is an edge of \mathcal{H} , then let \mathcal{H}_E be the hypergraph whose edge set consists of the minimal (under inclusions) members of $\{E' \cup E : E' \neq E \text{ is an edge of } \mathcal{H}\}$.

Lemma 3.3. *Let \mathcal{H} be a hypergraph with at least two edges, E be an edge of \mathcal{H} , and \mathcal{H}_E be as in the preceding paragraph. Then $\text{reg}(I(\mathcal{H})) \leq \max\{\text{reg}(I(\mathcal{H} \setminus E)), \text{reg}(I(\mathcal{H}_E)) - 1\}$.*

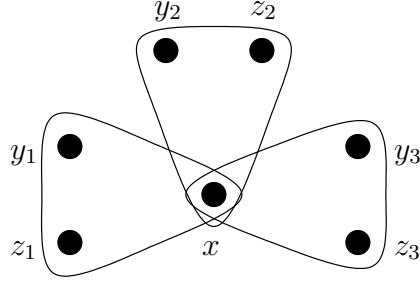
Lemma 3.3 follows from an easy long exact sequence argument, details of which can be found in [10, Theorem 6.2].

We are now ready to prove Theorem 1.2. We begin with the case where $c = 1$:

Lemma 3.4. *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph such that $\{E_0\}$ is a 2-collage for \mathcal{H} , then $\text{reg}(I(\mathcal{H})) = |E_0|$.*

Proof. We proceed by induction on the number of edges in \mathcal{H} . If \mathcal{H} has a single edge, then $\text{reg}(I(\mathcal{H})) = \text{reg}(x^{E_0}) = |E_0|$, as desired. Otherwise, let $E \neq E_0$ be any other edge of \mathcal{H} , and apply Lemma 3.3 to give that $\text{reg}(I(\mathcal{H})) \leq \max\{\text{reg}(I(\mathcal{H} \setminus E)), \text{reg}(I(\mathcal{H}_E)) - 1\}$. But then $\{E \cup E_0\}$ is a 2-collage for \mathcal{H}_E , and the result follows from induction together with the observation that $|E \cup E_0| = |E| + 1$. \square

Proof of Theorem 1.2. Let $\{E_1, \dots, E_c\}$ be a 2-collage, and for each i let \mathcal{H}_i be the simple hypergraph consisting of all E with $E \setminus \{v\} \subseteq E_i$ (as in the definition of 2-collage). By construction we have that $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^c \mathcal{E}(\mathcal{H}_i)$ and that each \mathcal{H}_i meets the conditions of Lemma 3.4. The result now follows from Theorem 3.1. \square

FIGURE 3.1. A 3-uniform hypergraph with regularity greater than $(3 - 1) \cdot \nu$.

Recall that a *matching* in a simple hypergraph \mathcal{H} is a collection of pairwise disjoint edges, and the *matching number* of \mathcal{H} (denoted by $\nu(\mathcal{H})$) is the maximum size of any matching in \mathcal{H} . We similarly denote the minimax matching numbers as $\nu_{\min}(\mathcal{H})$. In [10], it was essentially shown that if G is a graph then

$$\text{reg}(R/I(G)) \leq \nu_{\min}(G).$$

In the next example, we shall see that the analogous bound of $(d-1)\nu_{\min}(\mathcal{H})$ no longer holds for hypergraphs, thus that the 2-collage in the statement of Theorem 1.2 cannot be replaced by a matching (even in the uniform case).

Example 3.5. For $s > 1$, consider the hypergraph \mathcal{H}_s on the vertex set $\{x, y_1, \dots, y_s, z_1, \dots, z_s\}$ with edges $\{x, y_i, z_i\}$ (for $i = 1, \dots, s$). Figure 3.1 illustrates \mathcal{H}_3 . We have that the matching number and minimax matching number of \mathcal{H}_s are both 1. On the other hand, it is straightforward to compute that $\text{reg}(R/I(\mathcal{H}_s)) = s + 1$, which can be taken to be arbitrarily far from $(3 - 1)\nu(\mathcal{H}_s) = (3 - 1)\nu_{\min}(\mathcal{H}_s) = 2$.

It is easy to see that the bounds in Theorems 1.2 and 1.3 are not sharp. Indeed [25] observes that the disjoint union of cyclic graphs can give arbitrarily large differences between the induced matching number, regularity, and matching number.

We close this section by stating an equivalent form to Theorem 1.2 in somewhat different language. The definition of matching essentially calls for a set of edges that are as separated as possible. A notion of separation that allows us to interpolate between a hypergraph matching and an arbitrary set of edges is as follows:

Definition 3.6. Let t be a positive integer. Two distinct edges E and F are said to be *t-separated* if either $|E \setminus F| \geq t$ or $|F \setminus E| \geq t$.

Thus, a matching in a graph is a collection of pairwise 2-separated edges; more generally a matching in any d -uniform hypergraph is a collection of pairwise d -separated edges. It is immediate from definition that a 2-collage is exactly a maximal collection of pairwise 2-separated edges. (Indeed, one is immediately led to a definition of d -collage.)

Corollary 3.7. *Let \mathcal{H} be a simple hypergraph with edge ideal $I \subseteq R$, and let $\{E_1, \dots, E_c\}$ be a maximal collection of pairwise 2-separated edges in \mathcal{H} . Then $\text{reg}(R/I) \leq \sum_{i=1}^c (|E_i| - 1)$.*

4. REGULARITY IN A VERTEX-DECOMPOSABLE SIMPLICIAL COMPLEX

In this section, we will prove Theorem 1.5, and stronger versions thereof. Our main tool will be the combinatorial topology of simplicial complexes, together with the characterization of regularity from Lemma 2.2. Throughout this section and the next we will freely abuse notation to write $\text{reg } \Delta$ for $\text{reg}(R/I_\Delta)$.

As mentioned previously, a straightforward Mayer-Vietoris argument yields:

Lemma 4.1. [7, Lemma 2.10] *If \mathcal{H} is a simple hypergraph with edge ideal $I \subseteq R$, and v is any vertex of \mathcal{H} , then $\text{reg}(I)$ is either $\text{reg}(I : x) + 1$ or $\text{reg}(I, x)$.*

Restated in terms of the independence complex Δ of \mathcal{H} , Lemma 4.1 says that either $\text{reg } \Delta = \text{reg}(\text{link}_\Delta v) + 1$ or else $\text{reg } \Delta = \text{reg}(\text{del}_\Delta v)$. This is particularly intuitive from a geometric perspective, where it essentially says that either v is contained in an (induced) homology cycle of highest possible dimension, or else that some such homology cycle avoids v .

It follows immediately that

$$(4.1) \quad \text{reg } \Delta \leq \max\{\text{reg}(\text{link}_\Delta v) + 1, \text{reg}(\text{del}_\Delta v)\}.$$

It is clear from Lemma 2.2 part (2) that $\text{reg}(\text{del}_\Delta v) \leq \text{reg } \Delta$, hence, if the max of (4.1) is obtained on $\text{reg}(\text{del}_\Delta v)$, then $\text{reg } \Delta = \text{reg}(\text{del}_\Delta v)$. On the other hand, while Lemma 2.5 gives that $\text{reg } \Delta$ may not be smaller than $\text{reg}(\text{link}_\Delta v)$, taking Δ to be a cone with apex v over any complex shows that $\text{reg } \Delta$ may equal $\text{reg}(\text{link}_\Delta v)$, hence may be strictly less than $\text{reg}(\text{link}_\Delta v) + 1$.

Theorem 1.5 gives a concrete set of circumstances which guarantee that a homology n -cycle in $\text{link}_\Delta v$ lifts to a homology $(n+1)$ -cycle in Δ . We will prove the following generalization:

Theorem 4.2. *If v is a shedding vertex for a simplicial complex Δ such that $\Delta \setminus v$ is sequentially Cohen-Macaulay, then $\text{reg } \Delta = \max\{\text{reg}(\Delta \setminus v), \text{reg}(\text{link}_\Delta v) + 1\}$.*

We will prove Theorem 4.2 via several lemmas.

Lemma 4.3. *If σ is a face and v a shedding vertex of a simplicial complex Δ such that $v \notin \sigma$, then v is a shedding vertex for $\text{link}_\Delta \sigma$.*

Proof. Immediate from definition, since any face containing $\sigma \cup v$ satisfies the shedding vertex property. \square

We denote by $\Gamma^{[n]}$ the *pure n -skeleton* of a simplicial complex Γ , consisting of all faces contained in a face of dimension n .

The following lemma will be the core of the proof of Theorem 4.2.

Lemma 4.4. *Let Γ be a simplicial complex, and suppose that $\tilde{H}_n(\text{link}_\Gamma v) \neq 0$. If $(\text{link}_\Gamma v)^{[n]}$ is contained in a subcomplex Γ_0 of $\text{del}_\Gamma v$ with $\tilde{H}_n(\Gamma_0) = 0$, then $\tilde{H}_{n+1}(\Gamma) \neq 0$.*

Proof. We use the exactness of the Mayer-Vietoris sequence

$$\cdots \rightarrow \tilde{H}_{n+1}(\Gamma) \xrightarrow{g} \tilde{H}_n(\text{link}_\Gamma v) \xrightarrow{f} \tilde{H}_n(\text{del}_\Gamma v) \rightarrow \cdots.$$

The result then follows by noting that the map f is induced from the inclusion map. \square

Duval [8, Theorem 3.3] proved that a complex Δ is sequentially Cohen-Macaulay if and only if $\Delta^{[n]}$ is Cohen-Macaulay for all n . We use this to prove:

Corollary 4.5. *If $\text{del}_\Gamma v$ is a sequentially Cohen-Macaulay complex, and v is a shedding vertex of Γ with $\tilde{H}_n(\text{link}_\Gamma v) \neq 0$, then $\tilde{H}_{n+1}(\Gamma) \neq 0$.*

Proof. By definition of shedding vertex, $(\text{link}_\Gamma v)^{[n]}$ sits inside $\Gamma_0 = (\text{del}_\Gamma v)^{[n+1]}$. Then Γ_0 is Cohen-Macaulay of dimension $n+1$, hence $\tilde{H}_i(\Gamma_0) = 0$ for all $i \leq n$. The result follows by Lemma 4.4. \square

We take a brief aside to provide a more geometric proof of Corollary 4.5 when $\text{del}_\Gamma v$ satisfies the stronger condition of shellability. We recall that a shellable complex is built up by inductively attaching facets, in such a way that the homotopy type changes only when a facet is attached along its entire boundary. The following proposition will allow us to give a homotopy type version of Corollary 4.5 in this broad special case.

Proposition 4.6. *If $\text{del}_\Gamma v$ is a shellable complex and v is a shedding vertex of Γ , then $\text{link}_\Delta v$ sits inside a contractible subcomplex Γ_0 of $\text{del}_\Gamma v$.*

Proof. Let \mathcal{F} be the set of facets in the shelling of $\text{del}_\Gamma v$ which attach along their entire boundary, and let $\Gamma_0 = (\text{del}_\Gamma v) \setminus \mathcal{F}$. Since $\text{link}_\Delta v$ contains no facets of $\text{del}_\Gamma v$, we have that $\text{link}_\Delta v \subset \Gamma_0$. It is a standard fact in the theory of shellable complexes that Γ_0 is contractible – see e.g. [3, proof of Theorem 4.1], where Γ_0 is written with the notation Δ^* . \square

With just a little more work, we can recursively compute the homotopy type of Γ . Recall that the wedge product $X \wedge Y$ of two topological spaces X and Y is obtained by identifying some point in X with some point in Y , and let $\text{susp}(\Delta)$ denote the suspension of Δ .

Corollary 4.7. *If $\text{del}_\Gamma v$ is a shellable complex and v is a shedding vertex of Γ , then*

$$\Gamma \simeq (\text{del}_\Gamma v) \wedge \text{susp}(\text{link}_\Gamma v).$$

In particular $\tilde{H}_{n+1}(\Gamma) \cong \tilde{H}_{n+1}(\text{del}_\Gamma v) \oplus \tilde{H}_n(\text{link}_\Gamma v)$.

Proof. This follows immediately by [2, Lemma 10.4(ii)], taking $\Delta_0 = (\text{del}_\Gamma v) \setminus \mathcal{F}$ (as in the proof of Proposition 4.6), Δ_1 to be the complex generated by \mathcal{F} , and Δ_2 to be $v * (\text{link}_\Gamma v)$. \square

Remark 4.8. If in the statement of Corollary 4.7 we also have $\text{link}_\Gamma v$ to be shellable (as occurs when Γ is vertex-decomposable), then there is a proof avoiding the somewhat difficult gluing result [2, Lemma 10.4], and using instead more elementary results about homotopy type of

shellable complexes. For in this case, by [24, Lemma 6] Γ can be shelled by taking the shelling of $\text{del}_\Gamma v$ followed by that of $v * \text{link}_\Gamma v$. The special case then follows from [3, Theorems 3.4 and 4.1] and a straightforward computation.

We are now ready to prove our theorem:

Proof of Theorem 4.2. Suppose that $d = \text{reg}(\text{link}_\Delta v)$. Then by Lemma 2.2 there exists a face σ of $\text{link}_\Delta v$ such that

$$\tilde{H}_{d-1}(\text{link}_{\text{link}_\Delta v} \sigma) = \tilde{H}_{d-1}(\text{link}_\Delta(\sigma \cup v)) \neq 0.$$

Every link in a sequentially Cohen-Macaulay complex is also sequentially Cohen-Macaulay, so in particular $(\text{link}_\Delta \sigma) \setminus v = \text{link}_{\Delta \setminus v} \sigma$ is sequentially Cohen-Macaulay. By Corollary 4.5 we have that $\tilde{H}_d(\text{link}_\Delta \sigma) \neq 0$, hence that $\text{reg } \Delta \geq \text{reg}(\text{link}_\Delta v) + 1$, which suffices to prove the result. \square

We remark that the “dual” characterization of regularity, as in part (3) of Lemma 2.2, seems to be an essential part of the proof of Theorem 4.2, as it is much easier to understand the structure of links (versus induced subcomplexes) in a vertex-decomposable complex.

Corollary 4.9. *If Δ is a vertex-decomposable simplicial complex, then $\text{reg } \Delta$ can be recursively computed. If Δ has n vertices with $\dim \Delta = d$, then computing $\text{reg } \Delta$ requires computing the homology of at most $O(n^d)$ subcomplexes.*

Proof. The recursive algorithm is clear. The time bound is because computational paths involve making at most n choices between the link and deletion, of which at most d can be “link”. Hence the number of homology computations required is $O(\binom{n}{d}) = O(n^d)$. \square

The more natural problem from the algebra point of view would be: given a graph or hypergraph \mathcal{H} with edge ideal $I \subseteq R$, compute $\text{reg}(R/I)$. Unfortunately, this problem is NP-hard even for vertex-decomposable graphs, as can be seen by considering a whiskered graph [25, Section 4.5]. Since computing the independence complex of a graph is already an NP-hard problem, the question of whether $\text{reg } \Delta$ may be efficiently computed from an appropriate representation of Δ (e.g., a list of facets) appears to still be open in general. Corollary 4.9 settles this question in the affirmative for vertex-decomposable complexes of fixed dimension such that the shedding order can be efficiently computed.

5. A PACKING BOUND ON REGULARITY

In this section, we prove Theorem 1.6. We begin by observing:

Lemma 5.1. *For any simplicial complex Δ , we have*

$$\max\{\text{reg}(\text{link}_\Delta v) : v \in V(\Delta)\} \leq \text{reg } \Delta \leq \max\{\text{reg}(\text{link}_\Delta v) : v \in V(\Delta)\} + 1.$$

Proof. The lower bound is Lemma 2.5. To prove the upper bound, suppose that $\text{reg } \Delta = d$. Then, by Lemma 2.2 part (2), there is a subset S such that $\tilde{H}_{d-1}(\Delta[S]) \neq 0$. Without loss of generality, we can take S to be minimal under inclusion, so that $d = \text{reg } \Delta[S]$, but for any

proper subset $T \subset S$ we have $\text{reg } \Delta[T] < d$. Thus, for any $v \notin S$ we have $\text{reg}(\text{del}_{\Delta[S]} v) = \text{reg}(\Delta[S \setminus v]) < d$. By Lemma 4.1 it then follows that $d = \text{reg } \Delta[S] = \text{reg}(\text{link}_{\Delta[S]} v) + 1$. Since $\text{link}_{\Delta[S]} v = (\text{link}_{\Delta} v)[S]$, we get that $\text{reg}(\text{link}_{\Delta} v) \geq d - 1$, as desired. \square

In plain language, Lemma 5.1 says that we can find a vertex v of Δ such that the regularity of the link of v is at most one less than that of Δ .

Remark 5.2. Lemma 5.1 was shown for shellable complexes in [18, Theorem 2.4].

For a vertex v in simplicial complex Δ , we let the *degree* of v , denoted $\deg v$, be the degree of v in the corresponding hypergraph (i.e., the number of edges in $\mathcal{H}(\Delta)$ containing v). Equivalently, $\deg v$ is the number of minimal non-faces of Δ containing v . Thus, a vertex has degree zero if and only if it is contained in every maximal face, i.e., if and only if Δ is a cone over $\text{del}_{\Delta} v = \text{link}_{\Delta} v$.

Since degree 0 vertices can be deleted without affecting regularity, the next lemma follows immediately:

Lemma 5.3. *For any simplicial complex Δ , we can find a vertex v of non-zero degree such that $\text{reg } \Delta \leq \text{reg}(\text{link}_{\Delta} v) + 1$.*

By repeated applications of Lemma 5.3, we obtain:

Theorem 5.4. *For any simplicial complex Δ , we have $\text{reg } \Delta$ to be at most the maximum size of a minimal face σ with the property that $\text{link}_{\Delta} \sigma$ is a simplex.*

In the case where Δ is the independence complex of a graph, we can do somewhat better:

Lemma 5.5. *Suppose that G is a graph having no isolated edges, and let $\Delta = \Delta(G)$. Then we can find a vertex v with $\deg v > 1$ such that $\text{reg } \Delta \leq \text{reg}(\text{link}_{\Delta} v) + 1$.*

Proof. If G has no vertices of degree 1, then the result follows by Lemma 5.1. Otherwise, let x be a vertex of degree 1, and let y be the unique neighbor of x in G . We have $\text{link}_{\Delta} x = \Delta(G \setminus \{x, y\})$, hence

$$\text{reg}(\text{link}_{\Delta} x) = \text{reg}(\Delta(G \setminus x \setminus y)) = \text{reg}(\Delta(G \setminus y)) = \text{reg}(\text{del}_{\Delta} y),$$

where the second equality follows because x is an isolated vertex in $G \setminus y$. Then there are two possibilities:

Case 1. $\text{reg}(\text{del}_{\Delta} y) + 1 = \text{reg } \Delta$.

Then by Lemma 4.1 we have $\text{reg } \Delta = \text{reg}(\text{link}_{\Delta} y) + 1$, and we take $v = y$.

Case 2. $\text{reg}(\text{del}_{\Delta} y) = \text{reg } \Delta$.

Then we have $\text{reg } \Delta = \text{reg}(\text{del}_{\Delta} y) = \text{reg}(\Delta(G \setminus x \setminus y)) \leq \text{reg}(\text{del}_{\Delta} x) \leq \text{reg } \Delta$, and we can delete x from G without affecting regularity. If $G \setminus x$ has no isolated edge, then there is a vertex v with the desired properties by induction on the number of vertices.

If $G \setminus x$ has an isolated edge $\{z, w\}$, then $\text{reg } \Delta = \text{reg}(\text{del}_{\Delta} x) = \text{reg}(\text{link}_{\text{del}_{\Delta} x} z) + 1$, and so $\text{reg } \Delta \geq \text{reg}(\text{link}_{\Delta} z) \geq \text{reg}(\text{link}_{\text{del}_{\Delta} x} z)$. Thus, z (and by symmetry w)

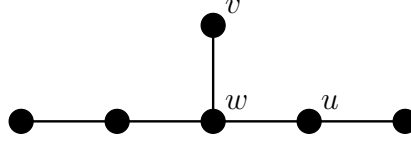


FIGURE 5.1. A graph G with $\zeta(G) = \text{reg}(I/R) = 2$, smaller than $\alpha(G) = 3$.

have the desired regularity property, and at least one of z and w has degree > 1 in G . \square

Proof of Theorem 1.6. Repeatedly apply Lemma 5.5, setting isolated edges aside as they are created in the graph. At the end, we have a set of vertices σ , each of degree at least 2, so that $\text{link}_{\Delta(G)} \sigma = \Delta(G \setminus N[\sigma])$ is the independence complex of a subgraph of G consisting of ℓ isolated edges together with some number of isolated vertices. In particular, $\text{reg}(\text{link}_{\Delta(G)} \sigma) = \ell$. Furthermore, Lemma 5.5 gives that $\text{reg}(R/I) = \text{reg} \Delta(G) \leq \text{reg}(\text{link}_{\Delta} \sigma) + |\sigma| = \ell + |\sigma|$.

Since the star at each vertex in σ is nondegenerate, we get a maximal center-separated star packing \mathcal{P} with $\zeta_{\mathcal{P}} = |\sigma| + \ell$, and so $\text{reg}(R/I) \leq \zeta_{\mathcal{P}} \leq \zeta(G)$. \square

Remark 5.6. The parameter $\zeta(G)$ is clearly at most the parameter $a'(G)$ used in [18, Theorem 2.1]; and Theorem 1.6 does not require vertex-decomposability, as their result does.

Example 5.7. It is instructive to examine the graph G pictured in Figure 5.1. Because G is chordal, $\text{reg}(R/I) = \nu_{\text{ind}}(G) = 2$ [10, Corollary 1.7]. Our star-packing invariant $\zeta(G)$ is 2 for this graph, achieved by taking the star at u (leaving an isolated edge). Thus, $\zeta(G) = \text{reg}(R/I)$ here. We remark that this example shows that the minimax version of $\zeta(G)$ (i.e., minimum size of a maximal star packing, plus the number of leftover edges) will not bound regularity, as taking a star at w will show.

The bound from Theorem 5.4 is on the other hand 3, as can be achieved by taking all the vertices of degree 1. Since the independence number $\alpha(G)$ is also 3, and since $\text{reg}(R/I)$ is obviously at most $\alpha(G)$, the latter bound is in this case trivial. There are situations where the bound from Theorem 5.4 is nontrivial: for example, if we expand G by adding a pendant at v , then the resulting graph K still has bound from Theorem 5.4 equal to 3, although $\alpha(K) = 4$.

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